

Hydrodynamic behaviour of an Abelian Sandpile Model with Laplacian rules

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We present a sandpile model, in which the instability of a site is determined also by the variables in a neighbourhood. This is a modification of the Abelian Sandpile Model, in which abelianity is preserved: it shares several mathematical properties of the original abelian model, while producing a more realistic dynamics. We show how our model presents interesting hydrodynamic features.

I. INTRODUCTION

Avalanche processes describe a variety of phenomena in Nature, ranging from, of course, avalanches (e.g. in piles of sand), to epidemic spread, allometric growth, . . . They are considered as a possible theoretical basis of $1/f$ noise (pink noise), and of the appearance of Self-Organised Criticality in physical and natural systems [1, 2].

A lattice modelisation of non-equilibrium avalanche dynamics is expected to be based on local *toppling* and *diffusion* rules: if a toppling threshold inequality is satisfied at x (i.e., x is *unstable*, e.g., the slope at x is too steep), a diffusion process occurs at x , which may trigger further topplings, and so on, producing an avalanche.

When more sites are simultaneously unstable, an ordering prescription must be given for definiteness. This produces a variety of (stochastic and deterministic) possible rules, that share the same crucial classification difficulties of non-equilibrium Statistical Mechanics, in contrast with the more clear and rigorous understanding of universality in equilibrium Critical Phenomena.

A simple dynamics, introduced by Bak, Tang and Wiesenfeld [1], involves a *fixed*, and *ultra-local* threshold rule, of the form $z(x) > h$ (instead of a *Laplacian* one, that involves heights $z(y)$ at neighbours $y \sim x$). Remarkably, as shown in the work of Dhar and collaborators [3, 4], for such a dynamics topplings at different sites *commute*, this justifying the name of *Abelian Sandpile Model* (ASM) given to this class of systems. The ingredient of abelianity is crucial in the determination of several hidden mathematical features of the model, among which, notably, a bijection with *spanning trees*, configurations in an *equilibrium* Statistical Mechanics model. Abelianity also allows to achieve scaling to the continuum limit, either by the forementioned correspondence, or by the fact that adding particles and relaxing in rounds is equivalent to add all the particles at once, and perform a unique complete relaxation, this latter process being easier to analyse theoretically in the large-volume limit (besides that algorithmically convenient).

Unfortunately, abelianity seems to be a fragile ingredient. Small modifications of threshold or diffusion rules

easily break commutativity. In particular, except for a construction specific to 1-dimensional chains, it seems impossible to implement genuine and realistic toppling rules, based on gradient or Laplacian, within ASM's. In [5] we presented a new class of ASM's, allowing for multi-site topplings: this increases the spectrum of possibilities, but still within ultra-local threshold rules.

Here we present a family of ASM's based on *Laplacian-like* toppling rules, possibly *non-linear*. The most natural rule should be linear homogeneous, e.g. of the form $z(x) > \langle z(y) \rangle_{y \sim x}$. However we need to introduce non-linearity, both for preserving abelianity, and for having a compact space of configurations. The simplest variant is of the form $z(x) > \alpha \langle z(y) \rangle_{y \sim x} + \delta$, thus linear inhomogeneous. Other variants are discussed later on.

Furthermore, in order to preserve abelianity, we need to adopt non-compact diffusion rules: if ordinary nearest-neighbour diffusion is $(Df)(x) = \sum_{y \sim x} (f(y) - f(x))$, we need a more general diffusion operator, of the form $(D_u f)(x) = \sum_y u(x - y)(f(y) - f(x))$. Fortunately, finite-range functions $u(r)$, e.g. $u(r) \sim \exp(-\lambda|r|)$, are allowed, and should show the same phenomenology of nearest-neighbour ones.

Interestingly, even in our largest class of models, configurations recurrent under a dynamics of random increases of the heights (plus relaxation), as in ordinary ASM's, still have an uniform steady-state probability distribution, and a generating function given by the Kirchhoff matrix-tree formula. It is possible that natural bijections exist with spanning trees also in our wider context.

Thus we have continuous-variable sandpiles with fixed-threshold (F) or non-local Laplacian-like (L) rules, and with compact (C) or non-compact (N) diffusion. Of the four possibilities, the three (FC), (FN) and (LN) have abelian realisations, with (FC) being the “ordinary” case. We will use the shortcuts X-, Y-, XY-ASM (X=F,L; Y=C,N) for these classes of models, and ASM for the original model with discrete variables.

In order to support our claim that Laplacian-like threshold rules are the crucial new ingredient here (while continuous variables and non-compact diffusion are a technical accident), we show how LN-ASM models have interesting phenomenological properties when observed under “hydrodynamic” experiments, that are not shared neither with the ordinary ASM and FC-ASM, nor with the FN-ASM models.

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II. THE MODEL

For simplicity we will only analyse translationally invariant sandpiles with continuum height variables, on a portion $V = \bigotimes_{1 \leq \alpha \leq d} \mathbb{Z}_{L_\alpha}$ of the d -dimensional hypercubic lattice, with periodic or open boundary conditions. Extensions, to be discussed elsewhere, could include models defined on arbitrary graphs, even directed, and realisations with discrete variables.

We start by considering dynamics in which the heights are *increased*, stochastically or deterministically, and avalanches are possibly produced. When we say that a set of configurations is “left stable by the dynamics”, we mean by any dynamics of this form. In particular, this excludes the inverse-toppling dynamics discussed in [5].

We denote by $\tilde{w}(\xi)$ the Laplace transform of $w(\mathbf{x})$. We say that a function $f(\mathbf{x}) : \mathbb{Z}^d \rightarrow \mathbb{R}$ is *symmetric* if it has the symmetries of the cubic lattice, $f(x_1, \dots, x_d) = f(\epsilon_1 x_{\sigma(1)}, \dots, \epsilon_d x_{\sigma(d)})$, for $\sigma \in \mathfrak{S}_d$ and $\epsilon \in \{\pm 1\}^d$.

Our sandpile model is determined by two non-negative symmetric functions $w(\mathbf{x})$ and $u(\mathbf{x})$, with $w(\mathbf{0}) = u(\mathbf{0}) = 0$, a dissipation parameter $\mu > 0$, and two functions $f(z)$, $g(z)$, strictly- and weakly-monotonic respectively, with $g(z)$ having a finite Lipschitz constant ℓ . We say that \mathbf{x} is *unstable* if

$$f(z(\mathbf{x})) > \sum_{\mathbf{y}} w(\mathbf{y}) g(z(\mathbf{x} + \mathbf{y})). \quad (1)$$

In such a case, a toppling $t_{\mathbf{x}}$ may occur at \mathbf{x} , modifying the height function as

$$t_{\mathbf{x}} : z(\mathbf{y}) \rightarrow \begin{cases} z(\mathbf{y}) + u(\mathbf{y} - \mathbf{x}) & \mathbf{y} \neq \mathbf{x} \\ z(\mathbf{x}) - (1 + \mu) \sum_{\mathbf{y}'} u(\mathbf{y}') & \mathbf{y} = \mathbf{x} \end{cases} \quad (2)$$

Note that $\mu > 0$ implies that $\sum_{\mathbf{x}} z(\mathbf{x})$ strictly decreases at each toppling. A configuration $z(\mathbf{x})$ is *stable* if no \mathbf{x} is unstable. We have the F-ASM if $f(z) = z$ and $g(z) = 1$. Furthermore, it is customary to take $w(\mathbf{x}) = u(\mathbf{x}) = \delta_{|\mathbf{x}|,1}$ the indicator functions on nearest neighbours, this producing a FC-ASM. We have an obvious affine covariance $z \rightarrow \gamma_1 z + \gamma_0$, with $\gamma_1 > 0$, and invariance under multiplication of the threshold rule (1) by a positive constant, that we exploit later on.

We require the sandpile to have three properties:

- [A] A positive cone $\Omega = \{z : z(\mathbf{x}) > z_{\min} \text{ for all } \mathbf{x}\}$ is left stable by the dynamics.
- [B] The set of stable configurations within Ω has finite non-zero measure.
- [C] Within Ω , the topplings are abelian.

Call $C = (1 + \mu) \sum_{\mathbf{y} \neq \mathbf{0}} u(\mathbf{y}) > 0$ and $\tilde{w}(\mathbf{0}) = \sum_{\mathbf{y}} w(\mathbf{y})$. We can use the covariance to fix $z_{\min} = 0$, i.e. $\Omega = (\mathbb{R}^+)^V$. In this case, condition [A] means that, if $z(\mathbf{y}) > 0$ for all \mathbf{y} , and \mathbf{x} is unstable, then (1) implies $z(\mathbf{x}) > C$, that is, using the monotonicity, $C \leq C' := f^{-1}(g(\mathbf{0})\tilde{w}(\mathbf{0}))$.

One easily sees that the set of stable configurations has non-zero measure, because $(0, C']^V$ is a subset. Suppose that the limit for $h \rightarrow +\infty$ of $f(h) - g(h)\tilde{w}(\mathbf{0})$ exists. If it is positive, there exists h_{\max} such that $f(h) > g(h)\tilde{w}(\mathbf{0})$ for all $h > h_{\max}$, and any configuration in Ω with $\max_{\mathbf{y}} z(\mathbf{y}) > h_{\max}$ is unstable at least at the position of the max, so stable configurations are contained within $(0, h_{\max}]^V$ and [B] is verified. If it is negative, there exists $\epsilon > 0$, and an unbounded set of h 's in \mathbb{R}^+ , such that the cubes $h - \epsilon < z(\mathbf{y}) < h$ are stable, because $f(h) \leq (g(h) - \epsilon\ell)\tilde{w}(\mathbf{0}) \leq g(h - \epsilon)\tilde{w}(\mathbf{0})$, and [B] is not verified. When the limit is equal to zero, or does not exist, the condition needs to be analysed more deeply, and we don't do this here.

A sufficient condition for [C], using the result on the analysis of [A] and the Lipschitzianity of $g(z)$, is that, for all $\mathbf{x} \neq \mathbf{0}$ and all $z > C'$,

$$f(z + u(\mathbf{x})) - f(z) > \ell \sum_{\mathbf{y} \neq \mathbf{0}} w(\mathbf{y}) u(\mathbf{x} - \mathbf{y}). \quad (3)$$

Assume that $f'(z) \geq \ell' > 0$ for all $z \geq C'$. Then we have the condition, for all $\mathbf{x} \neq \mathbf{0}$,

$$\ell' u(\mathbf{x}) \geq \ell (w * u)(\mathbf{x}). \quad (4)$$

It is easy to see that, if $u(\mathbf{x}) = u_0 \exp(-\lambda|\mathbf{x}|)$, then $\sup_{\mathbf{x} \neq \mathbf{0}} (w * u)(\mathbf{x})/u(\mathbf{x}) = \tilde{w}(\lambda\hat{\mathbf{n}})$ for $\hat{\mathbf{n}}$ some unit vector. For such a function u the condition reads $\tilde{w}(\lambda\hat{\mathbf{n}}) \leq \ell'/\ell$. From now on, we will only consider u 's of this form. Clearly $\tilde{w}(\lambda\hat{\mathbf{n}}) > \tilde{w}(\mathbf{0})$ for all $\lambda > 0$, and the difference goes to zero in the limit, thus some value $\lambda > 0$ satisfying the condition exists if and only if $\tilde{w}(\mathbf{0}) < \ell'/\ell$.

Use affine covariance to set $C = 1$. We can eliminate $\tilde{w}(\mathbf{0})$ to get the sufficient and necessary condition on f and g , to admit w , λ producing a LN-ASM

$$\frac{f(1)}{g(0)} < \min \left(\frac{\ell'}{\ell}, \lim_{z \rightarrow \infty} \frac{f(z)}{g(z)} \right). \quad (5)$$

Recall that $\ell' = \min_{z > C'} f'(z)$ and $\ell = \max_{z > 0} g'(z)$. One easily sees $f(M) \geq M\ell' + \mathcal{O}(1)$ and $g(M) \leq M\ell + \mathcal{O}(1)$, implying that the minimum above is always realised on the first quantity, i.e. the condition [C] is always stronger than [B]. In the case of a linear threshold rule, $g(z) = g_0 + zg_1$ and $f(z) = f_0 + zf_1$ (with $g_0, f_1 > 0$ and $g_1 \geq 0$) we get the constraint $(f_0 + f_1)g_1 < f_1g_0$. If $f(z)$ is quadratic, with $f_2 > 0$, we get analogously $(f_0 + f_1 + f_2)g_1 < (f_1 + 2f_2)g_0$.

We say that a realisation of sandpile model as above is *tight* if conditions [A] and [C] are satisfied in a tight way. Assuming that $f'(z)$ is monotonic, that $g(z) = g_0 + zg_1$ is linear, and setting $C = 1$, this means that w and λ are set to satisfy

$$\frac{f(1)}{g_0} = \tilde{w}(\mathbf{0}) < \tilde{w}(\lambda\hat{\mathbf{n}}) = \frac{f'(1)}{g_1}. \quad (6)$$

For $W \subseteq V$, define $\chi_W(\mathbf{x}) = 1$ if $\mathbf{x} \in W$ and 0 otherwise. It is easy to see that the frame identity Id_f

of the LN-ASM on V associated to (f, g, w, u) depends only on u and the domain V (in particular, it is the same as in the associated FN-ASM), and is given by $\text{Id}_f(\mathbf{x}) = -(\chi_V * u)(\mathbf{x})$, the difference in height after one toppling has occurred at each site. Conversely, the recurrent identity Id_r depends also on w, f and g .

III. MULTIPLE THRESHOLD RULES

For fixed w and u , we may have families of LN-ASM, which provide a continuous deformation of the FN-ASM with threshold rule $z(\mathbf{x}) \geq h$, and thus $f(z) = z$ and $g(z) = h/\tilde{w}(\mathbf{0})$.

We may extend the formalism of the previous section to functions $f, g : \mathbb{R} \rightarrow \mathbb{R}^s$, for $s \geq 1$, and, in (1), replace $>$ with \succ , the canonical partial ordering ($f \succ g$ iff $f_\alpha \geq g_\alpha$ for all $1 \leq \alpha \leq s$). This provides a different mechanism for continuous deformations of F-ASM's: we can take $s \geq 2$, and keep the original rule for one of the components.

In the analysis of properties [A] and [C], the different components have a non-trivial interplay in a unique respect. For property [A] we need $C \leq C'_\alpha := f'_\alpha(g_\alpha(0)\tilde{w}(\mathbf{0}))$, for all α . Then, for property [C] we need $\tilde{w}(\lambda\hat{\mathbf{n}}) \leq \ell'_\alpha/\ell_\alpha$, for all α . Apparently, these constraints are factorised, but this is not the case. While ℓ_α , the Lipschitz constant of $g_\alpha(z)$, is in fact independent from the other components, ℓ'_α is defined in terms of the allowed range for unstable heights, more precisely

$$\ell'_\alpha := \inf_{z > \max_\beta (C'_\beta)} (f'_\alpha(z)). \quad (7)$$

Say that the (one or more) indices realising the max of C'_β are the *leading components*. For a valid realisation of the sandpile in dimension $s \geq 2$, the restriction of the threshold equation to a subset of components containing at least one leading component still produces a valid realisation. If we require non-redundancy, we get the constraint

[D] For each $1 \leq \alpha \leq s$, there exists $z \in \Omega$ such that $f_\beta(z(\mathbf{x})) - \sum_{\mathbf{y}} w(\mathbf{y})g_\beta(z(\mathbf{y})) < 0$ only for $\beta = \alpha$.

This constraint is specially important if we have $s = 2$, $\alpha = 1$ is the fixed-threshold constraint $z(\mathbf{x}) > h$, and is also the only leading component. Our sandpile model is genuinely distinct from the original version if and only if the condition is verified, and the point at which this happens in a tight way is the starting point of the family of continuous deformations.

IV. ANALYSIS OF EXAMPLES IN $d=1$ AND 2

The tight realisations of the sandpile on an infinite linear chain, with $w(\mathbf{x}) = w\delta_{|\mathbf{x}|,1}$ and $u(\mathbf{x}) = uA^{-|\mathbf{x}|}$, require $A > 1$ and give

$$\frac{A^2 + 1}{2A} = \frac{f'(1)g_0}{f(1)g_1}, \quad u = \frac{2}{A-1}, \quad w = \frac{f(1)}{2g_0}. \quad (8)$$

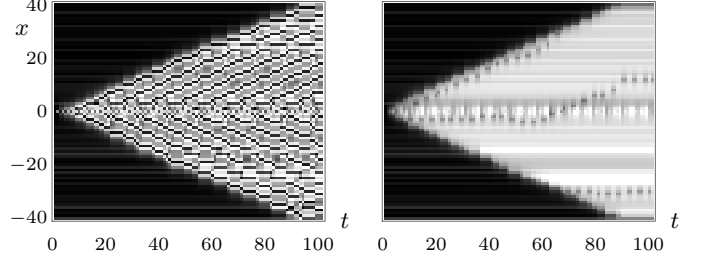


FIG. 1: Time evolution of a centrally-seeded sandpile in $d = 1$, with random low-height initialisation. Right: in a LN-ASM as described in Section IV, with $A = 2$. Left: in the associated FN-ASM. In all our figures, black/white = low/high.

If also $f(z)$ is linear, then we can set

$$f(z) = z - \frac{(A-1)^2}{A^2 + 1}, \quad g(z) = z + 1, \quad (9)$$

$$u = \frac{2}{A-1}, \quad w = \frac{2A}{A^2 + 1}. \quad (10)$$

The maximal possible height is $h_{\max} = (A^2 + 1)/(A-1)^2$.

Figure 1 compares the time trace of a simple protocol (steady injection of sand in the middle of the lattice), between the model above, for $A = 2$, and the FN-ASM using the same function $u(\mathbf{x})$. In both cases we observe a linear growth of a plateau. While in the F-ASM the heights in the plateau show fuzzy short-range fluctuations, in the L-ASM the profile is smooth, with no need of coarse-graining.

In the two-dimensional case, we can still choose the model to be tight, $w(\mathbf{x}) = w\delta_{|\mathbf{x}|,1}$, $g(z) = z + 1$, $f(z) = z + f_0$, $u(\mathbf{x}) = u\exp(-\lambda|\mathbf{x}|)$, and $C = C' = 1$ (thus $f_0 < 0$, $w = (1+f_0)/4$ and $\cosh(\lambda) = \frac{2}{w} - 1$). As the series $\mathcal{N}(\lambda) := \sum_{\mathbf{x} \neq \mathbf{0}} \exp(-\lambda|\mathbf{x}|)$ has no closed form, the parameter u has to be set numerically to $u = \mathcal{N}(\lambda)^{-1}$. For example, choosing $f_0 = -0.15$ gives $u = 0.1164...$ This is the realisation presented in the example of Figure 2, and later on in the following section. Again, we observe smoothness properties only in the LN-ASM case.

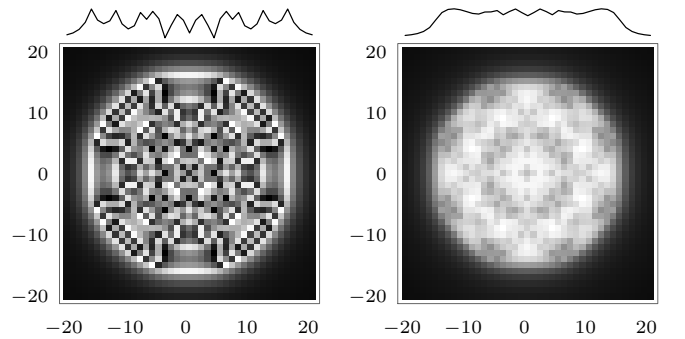


FIG. 2: Relaxation of $z(\mathbf{x}) = 32/(1 + |\mathbf{x}|^2)$ in the FN-ASM (left), and of $z(\mathbf{x}) = 196/(1 + |\mathbf{x}|^2)$ in the LN-ASM (right). On top: the profile of the middle row.

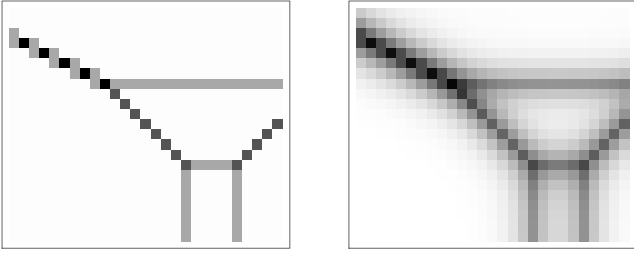


FIG. 3: Left: a portion of a typical outcome of toppling-antitoppling dynamics in the ASM. Right: the analogous outcome in our LN-ASM.

V. EXTREME REGIMES

We have seen above how the sandpiles with Laplacian-like toppling rules, when investigated in regimes imitating realistic experimental settings (“hydrodynamic regimes”), show smoothness properties not shared with the ordinary versions. These settings are characterised by the presence of empty regions, collecting the outcome of avalanches, and making them not percolating. In this section we discuss how the system behaves when it is driven towards extreme regimes, in which the avalanche size is regularised only by the finiteness of the volume. In this case, the new model preserves some resemblance with the intriguing combinatorial features of the ordinary ASM.

A natural candidate is the study of the *recurrent identity*, e.g. on a $L \times L$ portion of the square lattice, with open boundary condition. This configuration has a central role in the mathematical analysis of the model (it is the identity of the abelian group associated to recurrent configurations) [6]. It is obtained through a non-local procedure, as the fixed point of iterated addition of frame identities, and is very sensitive to the shape of the domain and boundary conditions. In the ordinary discrete ASM, it shows convergence to self-similar limit shapes, involving fractal Sierpinski-like structures. An appropriate discussion is too complex to fit here, and we start instead with analysing its “building blocks”, the *strings* [7]. These are one-dimensional structures in ASM’s on bidimensional lattices, that also play a role in the classification of the emerging periodic patterns under deterministic protocols [8, 9].

In the ordinary (discrete) ASM, strings first appear when performing “toppling-antitoppling” operations [5,

7] on the maximally-filled configuration ($z^{(\max)}(\mathbf{x}) = 3$ for all \mathbf{x} , on the square lattice). For LN-ASM’s we first need to generalise the concept of $z^{(\max)}(\mathbf{x})$, which turns out to be non-homogeneous. It is characterised as the unique solution to the system in which the threshold inequalities (1) are replaced by equalities. Thus, it is stable, but $z^{(\max)} + \epsilon \chi_{\{\mathbf{x}\}}$ is unstable for all $\mathbf{x} \in V$, $\epsilon > 0$.

If f, g are linear and w is nearest-neighbour, the corresponding system has the form $(\Delta + a)z^{(\max)}(\mathbf{x}) = b(\mathbf{x})$, plus vanishing boundary conditions. In our geometry, this is easily solved in Fourier basis and by method of images: the solution is smooth, and near to h_{\max} , the maximal possible height, for sites far from the boundary.

Next, we need to define toppling-antitoppling operators $\nu_{\mathbf{x}}$ in continuum ASM’s. Call z' the relaxation of $z + s\chi_{\{\mathbf{x}\}}$, where s is an amount making \mathbf{x} barely unstable. Then call z'' the antirelaxation of $z' - s\chi_{\{\mathbf{x}\}}$. We set $\nu_{\mathbf{x}}z := z''$. The action of continuous toppling-antitoppling operators $\nu_{\mathbf{x}}$ on $z^{(\max)}$, in LN-ASM, is analogous to the action of discrete toppling-antitoppling operators $a_{\mathbf{x}}^\dagger a_{\mathbf{x}}$ in ASM. Examples are shown in Figure 3.

The reason is that strings correspond to discontinuities in piecewise-linear toppling matrices, and even in continuous sandpiles the toppling matrices are discrete. This fact combines with the observations in [7, 9].

An aspect of this resemblance of ASM and LN-ASM occurs also in a conceptually-simpler protocol: add sand to an uniform configuration $z(\mathbf{x}) = h_{\max}$, at a unique site, then relax. Typical outcomes are shown in Figure 4.

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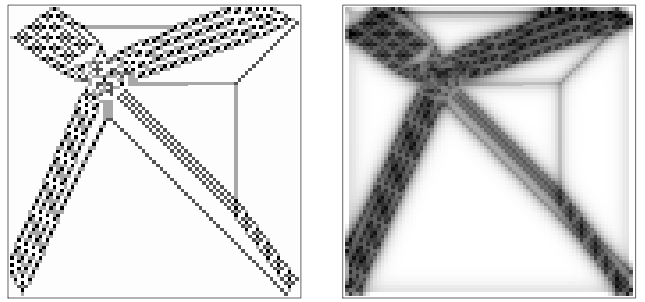


FIG. 4: Relaxing κL particles added at $\mathbf{x} = (L/3, L/4)$ to $z(\mathbf{x}) = h_{\max}$, on a $L \times L$ square (here $L = 81$). Left: ASM, and $\kappa = 2$. Right: GN-ASM, and $\kappa = 4$. Graytones are scaled by $z^{(\max)}(\mathbf{x})$.

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- [1] P. Bak, C. Tang and K. Wiesenfeld, PRL **59**, 381 (1987); Phys. Rev. A **38** 364 (1988).
 - [2] B. Chopard and M. Droz, *Cellular Automata modeling of physical systems*, Cambridge Univ. Press, 1998.
 - [3] D. Dhar, PRL **64** 1613 (1990); Physica A **263**, 4 (1999); S.N. Majumdar and D. Dhar, J. Phys. A **24** L357 (1991); Physica A **185** 129 (1992).
 - [4] D. Dhar, P. Ruelle, S. Sen and D.-N. Verma, J. Phys. A **28**, 805 (1995).
 - [5] S. Caracciolo, G. Paoletti and A. Sportiello, EPJ-ST (accepted), arXiv:1112.3491.
 - [6] M. Creutz, Comp. Phys. **5**, 198 (1991).
 - [7] S. Caracciolo, G. Paoletti and A. Sportiello, EPL **90**, 60003 (2010).
 - [8] S. Ostojic, Physica A **318**, 187 (2003).
 - [9] D. Dhar, T. Sadhu and S. Chandra, EPL **85**, 48002 (2009).